

One-boundary Temperley-Lieb algebras in the XXZ and loop models

A. Nichols¹, V. Rittenberg², and J. de Gier³

^{1,2}*Physikalisches Institut der Universität Bonn,
Nussallee 12, 53115 Bonn, Germany.*

³*Department of Mathematics and Statistics,
University of Melbourne, Parkville,
Victoria VIC 3010, Australia.*

Abstract

We give an exact spectral equivalence between the quantum group invariant XXZ chain with arbitrary left boundary term and the same XXZ chain with purely diagonal boundary terms.

This equivalence, and a further one with a link pattern Hamiltonian, can be understood as arising from different representations of the one-boundary Temperley-Lieb algebra. For a system of size L these representations are all of dimension 2^L and, for generic points of the algebra, equivalent. However at exceptional points they can possess different indecomposable structures.

We study the centralizer of the one-boundary Temperley-Lieb algebra in the ‘non-diagonal’ spin- $\frac{1}{2}$ representation and find its eigenvalues and eigenvectors. In the exceptional cases the centralizer becomes indecomposable. We show how to get a truncated space of ‘good’ states. The indecomposable part of the centralizer leads to degeneracies in the three mentioned Hamiltonians.

¹nichols@th.physik.uni-bonn.de

²vladimir@th.physik.uni-bonn.de

³degier@ms.unimelb.edu.au

1 Introduction

The one-dimensional anisotropic spin- $\frac{1}{2}$ Heisenberg model (the XXZ quantum chain) is not only a paradigm for integrable systems but is also an interesting model for describing experimental data [1]. Recently it was shown that for the special value of the anisotropy parameter $\Delta = -\frac{1}{2}$ the same Hamiltonian gives the time evolution of fluctuating interfaces, known as the raise and peel models [2,3]. Moreover in this case the ground-state wavefunctions for various boundary conditions have remarkable combinatorial properties [4,5].

In the present paper we will present some new properties of the XXZ chain with diagonal and non-diagonal boundary conditions. In particular we shall give a spectral equivalence between the XXZ chain with different types of boundary conditions.

In the diagonal case, after the pioneering work of Alcaraz et al. [6,7] on the Bethe ansatz, many properties are known. In particular, for a special choice of the boundary terms, the quantum chain has the quantum symmetry $U_q(SU(2))$ [8]. For this case an alternative understanding of the properties of the chain can be obtained from an algebraic point of view as the terms appearing in the Hamiltonian are the generators of the Temperley-Lieb (TL) algebra [9,10]. In this way, for example, it was shown that the spectra of the Potts models were contained in those of the XXZ chain [11,12]. Away from the $U_q(SU(2))$ boundary conditions, some degeneracies were observed numerically [13,14] but no explanation was found.

The case of two general non-diagonal boundaries has received a lot of attention more recently. Although it is well known to be integrable [15] there is no obvious Bethe reference state. At the decoupling point ($\Delta = 0$) the spectrum and wavefunctions have been obtained [16]. Away from this point, in the case in which the parameters satisfy an additional constraint, the Bethe ansatz equations were obtained using two different approaches. In the first approach the Bethe ansatz was constructed directly using the quantum chain [17–20]. In a second, completely different approach, an ‘equivalent’ Hamiltonian was written in the vector space of link patterns [21]. In the two approaches one begins with a Hamiltonian written in terms of generators of the two-boundary Temperley-Lieb algebra (2BTL) [5] and uses two different representations of the algebra - one acting in a spin basis and the other acting in a link pattern basis. As we shall explain one expects that although two Hamiltonians may have the same spectrum they may possess different Jordan cell structures making the number of eigenfunctions and the physics very different (Appendix A). These aspects of the problem were not studied in [17–21].

In the presentation of our work, we start in section 2 by introducing the one-boundary Temperley-Lieb algebra (1BTL), also known as the blob algebra, which is well known in the mathematical literature [22–25]. The 1BTL algebra depends on two parameters, one for the bulk generators and one for the boundary generator. In the cases we shall call ‘critical’ (as coined in [22–25]) when a simple relation between these two parameters is satisfied, the algebra becomes non-semisimple, and has representations

which are reducible but indecomposable.

We start with the ‘master’ Hamiltonian H^M which is composed of $L - 1$ bulk generators and one boundary generator chosen by convention to be at the left end of the 1BTL. This Hamiltonian has one explicit parameter (the coefficient of the boundary generator) in addition to the two parameters of the 1BTL. We first use the L -site ‘non-diagonal’ representation of the 1BTL defined in the 2^L dimensional spin basis. The one-boundary Hamiltonian is obtained by adding a general 2×2 matrix to the end of the quantum group invariant Hamiltonian. It should be remembered that the quantum group invariant Hamiltonian already has very particular diagonal terms at either end. The one-boundary Hamiltonian H^{nd} depends explicitly on three parameters and has no local conserved charge. We show that the spectrum, including degeneracies, of this Hamiltonian is identical with that of another XXZ Hamiltonian H^d defined in the same spin basis with general diagonal boundary terms. We consider this observation as one of the most important results of this paper. The relation between the spectra of the two Hamiltonians using the Bethe ansatz is given in Appendix E. We show, for the case of 2 generators, that the 1BTL has also possesses a ‘diagonal’ representation in the spin basis (for more on this topic see Appendix C).

In Section 3 we introduce the representation of 1BTL algebra acting on the space of link patterns (there are actually 2^L of them!) and obtain the Hamiltonian H^{lp} . The link patterns have a diagrammatic charge with the same spectrum as S^z of the quantum chain. However it is important to stress that this diagrammatic charge is not a conserved quantum number except in an ideal of the 1BTL corresponding to the charge zero sector.

The Hamiltonian H^{lp} has a lower block triangular structure. In order to obtain its spectrum, one can disregard the off-diagonal blocks and obtain a ‘fake’ Hamiltonian which commutes with the diagrammatic charge [21]. Although this Hamiltonian H^{lp} has the same spectrum as H^{nd} and H^d (see Appendix E), the indecomposable structures which can occur are different. Some simple examples are given to illustrate this point. In Appendix B we show in the case of 2 sites, the similarity transformations which relate the three Hamiltonians away from the ‘critical’ cases. The three representations of the 1BTL algebra, namely those in H^{nd} , H^d and H^{lp} , are all of dimension 2^L and the issue of their faithfulness is discussed in Appendix D.

The problem of integrable quantum field theory on the half-line has led to the discovery of ‘boundary quantum groups’ [26, 27] generalizing earlier results at free fermion point [28]. Based on these results Doikou [29] has found a centralizer of the 1BTL in the spin- $\frac{1}{2}$ representation for the finite chain. This operator is not related to the integrability of the model since it commutes with each member of the 1BTL algebra. In Section 4 we study the eigenvalues and the eigenfunctions of this centralizer. For generic values of the parameters of the 1BTL it is completely diagonalizable and its spectrum and degeneracies are related to that of an S^z -type charge. However, for the cases in which the 1BTL algebra is ‘critical’, this centralizer is not fully diagonalizable.

This explains the appearance (see Appendix A) of degeneracies in the spectrum of H^{nd} and, via the spectral equivalence, also in H^d and H^{lp} . Moreover the eigenfunctions of the centralizer allow a construction of the ‘good’ representations of the 1BTL algebra. This construction is based on truncated Bratelli diagrams and the results are closely related to previous work by Martin et al. [22–25].

In a similar way to the TL algebra, the 1BTL algebra also has Potts representations which correspond to Potts models with boundary terms. These models will be considered elsewhere [30]. Conclusions and open questions will be presented in Section 5.

2 The one-boundary Temperley-Lieb algebra and XXZ quantum chains

We would like to remind the reader of some known facts about the connections between the Temperley-Lieb algebra and the XXZ quantum chain [10]. The Temperley-Lieb algebra (TL) is an associative algebra with the generators e_i ($i = 1, \dots, L-1$) obeying the relations:

$$\begin{aligned} e_i e_{i\pm 1} e_i &= e_i \\ e_i e_j &= e_j e_i \quad |i - j| > 1 \\ e_i^2 &= (q + q^{-1}) e_i \end{aligned} \tag{2.1}$$

This has a representation in terms of Pauli matrices:

$$e_i = \frac{1}{2} \left\{ \sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y - \cos \gamma \sigma_i^z \sigma_{i+1}^z + \cos \gamma + i \sin \gamma (\sigma_i^z - \sigma_{i+1}^z) \right\} \tag{2.2}$$

with $q = e^{i\gamma}$. Using these generators we can define the quantum group invariant Hamiltonian H^{qg} of a ferromagnetic quantum chain:

$$\begin{aligned} H^{qg} &= - \sum_{i=1}^{L-1} e_i \\ &= - \frac{1}{2} \left\{ \sum_{i=1}^{L-1} \left(\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y - \cos \gamma \sigma_i^z \sigma_{i+1}^z + \cos \gamma \right) + i \sin \gamma (\sigma_1^z - \sigma_L^z) \right\} \end{aligned} \tag{2.3}$$

This integrable Hamiltonian, defined on the 2^L dimensional vector space, has an anisotropy parameter $\Delta = -\cos \gamma$ and is $U_q(SU(2))$ symmetric [8]. As is well known [8], if q is a root of unity the quantum group has ‘good’ (irreducible) representations as well as ‘bad’ (indecomposable) representations. As a consequence of the existence of ‘bad’ representations (see Appendix A), the spectrum of H^{qg} when q is a root of unity has higher degeneracies than occur for generic q . In this case it also has indecomposable structure (the number of eigenfunctions is smaller than 2^L).

The one-boundary Temperley-Lieb algebra (1BTL) is obtained [22–25] by adding a new generator e_0 to the TL algebra. It has the following additional relations:

$$\begin{aligned} e_1 e_0 e_1 &= e_1 \\ e_0^2 &= \frac{\sin \omega}{\sin(\omega + \gamma)} e_0 \\ e_0 e_i &= e_i e_0 \quad i > 1 \end{aligned} \tag{2.4}$$

Notice that in addition to the bulk parameter γ the 1BTL algebra has a second parameter ω which is only defined up to a multiple of π . These can both be complex in general but for convenience, we will assume that they are real in order to use trigonometric functions.

It was observed by Martin et al. [22–25] that if:

$$\omega = k\gamma + \pi\mathbf{Z} \tag{2.5}$$

with k integer, then the 1BTL algebra becomes non-semisimple and possesses indecomposable representations. We shall keep the rather unusual name ‘critical’ introduced in [24, 25]. The case $\omega = -\gamma$, which is also ‘critical’, requires a rescaling of the generator e_0 in (2.4). One should note that the indecomposability is controlled by ω and it can be ‘critical’ (called simple critical in [22–25]) even when γ is generic i.e. $q = e^{i\gamma}$ generic, as long as the relation (2.5) is satisfied.

Using the generators of the 1BTL algebra, we consider the ‘Master’ Hamiltonian:

$$H^M = -ae_0 - \sum_{i=1}^{L-1} e_i \tag{2.6}$$

where a is an arbitrary parameter.

As we are going to see, due to the existence of several different representations of the 1BTL algebra the properties of this Hamiltonian in different 2^L dimensional vector spaces are different. The spectra are the same but the Jordan cell structures are different.

The first representation of this algebra, which we shall call ‘non-diagonal’, is obtained taking the bulk e_i with $1 \leq i \leq L-1$ as in (2.2) and:

$$e_0 = -\frac{1}{2\sin(\omega + \gamma)} (i \cos \omega \sigma_1^z + \cos \phi \sigma_1^x + \sin \phi \sigma_1^y - \sin \omega) \tag{2.7}$$

The angle ϕ is irrelevant as it can be changed by a rotation of σ_1^x and σ_1^y preserving the bulk generators (2.2). In this paper we shall put $\phi = 0$. Then we have:

$$\begin{aligned} e_0 &= -\frac{1}{2\sin(\omega + \gamma)} (i \cos \omega \sigma_1^z + \sigma_1^x - \sin \omega) \\ &= -\frac{1}{2\sin(\omega + \gamma)} \begin{pmatrix} ie^{i\omega} & 1 \\ 1 & -ie^{-i\omega} \end{pmatrix} \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1} \end{aligned} \tag{2.8}$$

In this way we obtain:

$$H^{nd} = \frac{\sin \gamma}{\cos \omega + \cos \delta} (i \cos \omega \sigma_1^z + \sigma_1^x - \sin \omega) - \frac{1}{2} \left\{ \sum_{i=1}^{L-1} (\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y - \cos \gamma \sigma_i^z \sigma_{i+1}^z + \cos \gamma) + i \sin \gamma (\sigma_1^z - \sigma_L^z) \right\} \quad (2.9)$$

where we have used the following convenient parameterization for a :

$$a = \frac{2 \sin \gamma \sin(\omega + \gamma)}{\cos \omega + \cos \delta} \quad (2.10)$$

Notice that H^{nd} is dependent on three parameters: γ and ω , related to the algebra and δ related to the constant a in the Hamiltonian (2.6). We shall see that their roles in the physical properties of H^{nd} are different.

It is interesting to observe that on the first site, H^{nd} contains the most general boundary term. This is not the case for the last site where H^{nd} has the same boundary term as H^{qg} c.f. (2.3) and (2.9). One can see that there is no local charge which commutes with H^{nd} .

One of the main results of this paper is that for any values of the three parameters, the spectrum of H^{nd} exactly coincides with the spectrum of the Hamiltonian H^d with diagonal boundary terms only:

$$H^d = -\frac{1}{2} \left\{ \sum_{i=1}^{L-1} (\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y - \cos \gamma \sigma_i^z \sigma_{i+1}^z + \cos \gamma) + \sin \gamma \left[\tan \left(\frac{\omega + \delta}{2} \right) \sigma_1^z + \tan \left(\frac{\omega - \delta}{2} \right) \sigma_L^z + \frac{2 \sin \omega}{\cos \omega + \cos \delta} \right] \right\} \quad (2.11)$$

We postpone for a moment the proof of this statement. Unlike H^{nd} , the diagonal chain has the obvious local charge:

$$S^z = \frac{1}{2} \sum_{i=1}^L \sigma_i^z \quad (2.12)$$

The fact that the spectra are identical does not necessarily mean that in the whole parameter space, one can construct a similarity transformation relating these two Hamiltonians. In order to illustrate this point, in Appendix B we derive the similarity transformation which relates H^{nd} to H^d for two sites. The similarity transformation exists everywhere except for $\omega = -\gamma, 0, \gamma$. For a larger number of sites there are many more possible values of ω where a similarity transformation would break down. At these points the two Hamiltonians have repeated eigenvalues and different Jordan cell structures. In many examples, by exact diagonalization, we found that H^{nd} always has Jordan cell structures for these cases and H^d , being Hermitian at least for real values of the parameters γ, ω and δ , does not.

$$\begin{array}{|c|} \hline \cdot \cdot \cdot \\ \hline \cdot \cdot \cdot \\ \hline \end{array} = \frac{\sin \omega}{\sin(\omega+\gamma)} \begin{array}{|c|} \hline \cdot \cdot \cdot \\ \hline \end{array} \quad (3.4)$$

$$\begin{array}{|c|} \hline \text{---} \\ \hline \text{---} \\ \hline \text{---} \\ \hline \text{---} \\ \hline \end{array} = \begin{array}{|c|} \hline \text{---} \\ \hline \text{---} \\ \hline \end{array} \quad (3.5)$$

The link pattern representation corresponds to considering an ideal of the 1BTL. We consider the state $|I\rangle$ by taking the graph corresponding to the unit element $\mathbf{1}$ of the 1BTL algebra. It has all L sites unconnected:

$$|I\rangle = \begin{array}{|c|} \hline \cdot \cdot \cdot \\ \hline \end{array} \quad (3.6)$$

We then act with the algebra and finally keep only the bottom half of the picture:

$$\mathbf{1}|I\rangle = \begin{array}{|c|} \hline \cdot \cdot \cdot \\ \hline \end{array} \quad (3.7)$$

$$e_i |I\rangle = \begin{array}{|c|} \hline \cdot \cdot \cdot \\ \hline \end{array} \begin{array}{|c|} \hline \cdot \cdot \cdot \\ \hline \end{array} \quad (3.8)$$

$$e_0 |I\rangle = \begin{array}{|c|} \hline \cdot \cdot \cdot \\ \hline \end{array} \quad (3.9)$$

Other examples are:

$$\begin{aligned} e_{i+1}e_i |I\rangle &= \begin{array}{|c|} \hline \cdot \cdot \cdot \\ \hline \cdot \cdot \cdot \\ \hline \end{array} \begin{array}{|c|} \hline \cdot \cdot \cdot \\ \hline \end{array} \\ &= \begin{array}{|c|} \hline \cdot \cdot \cdot \\ \hline \end{array} \begin{array}{|c|} \hline \cdot \cdot \cdot \\ \hline \end{array} \end{aligned} \quad (3.10)$$

and:

$$e_{i+1}e_{i+2}e_i |I\rangle = \begin{array}{|c|} \hline \cdot \cdot \cdot \\ \hline \end{array} \begin{array}{|c|} \hline \cdot \cdot \cdot \\ \hline \end{array} \quad (3.11)$$

C =Number of connections	Loop Diagrams	S^z charge
0		1
1)	-1
2	() ;))	0

Table 1: Diagrams for $L = 2$ sites

C =Number of connections	Loop Diagrams	S^z charge
0		$\frac{3}{2}$
1)	$-\frac{3}{2}$
2)) ; () ; ()	$\frac{1}{2}$
3))) ; ()) ;)()	$-\frac{1}{2}$

Table 2: Diagrams for $L = 3$ sites

Instead of the graphical representation we use a more convenient typographical notation. If a site i is connected to the left or right we write “)” and “(” respectively. If it is unconnected then we write “|”. Thus the picture (3.11) is written “ $|\cdots|(())|\cdots|$ ”. As we are dealing here with the 1BTL we *cannot* have links to the right boundary.

It is natural to introduce a diagrammatic charge, C , which counts the number of sites which contain links connected to another site or to the boundary. The link patterns form a vector space of dimension 2^L and the diagrammatic charge C splits it into subspaces having the dimension of the binomial coefficients. More specifically for a system of size L we have $C = 0, 1, \dots, L$ with:

- C even

$$\text{dimension} = \binom{L}{\frac{L}{2} - C} \quad (3.12)$$

- C odd

$$\text{dimension} = \binom{L}{-\frac{L}{2} + C - 1} \quad (3.13)$$

This is illustrated for the cases of 2 and 3 sites in Tables 1 and 2. It is easily seen that the action of the generators on any picture can never decrease the number of connections and this fact allows us to block diagonalize the Hamiltonian. Let us illustrate with the simple example at $L = 2$ sites.

Acting on the basis:

$$\begin{pmatrix} || \\)| \\ () \\)) \end{pmatrix} \quad (3.14)$$

we have:

$$e_0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & \frac{\sin \omega}{\sin(\omega+\gamma)} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \frac{\sin \omega}{\sin(\omega+\gamma)} \end{pmatrix} \quad (3.15)$$

$$e_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 2 \cos \gamma & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (3.16)$$

In Appendix D we show that, in the case of 2 generators, for generic values of γ and ω this representation is faithful. However at some subset of the ‘critical’ points (2.5) there exist additional relations between the words of the 1BTL and it is no longer faithful. We believe this is also true for the general case.

In the link pattern representation the eigenstates of the diagrammatic charge are of course the individual diagrams. The charge in this basis is therefore given by:

$$C = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \quad (3.17)$$

The Hamiltonian (2.6) is given by:

$$H^{lp} = -ae_0 - e_1 = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ -a & -a\frac{\sin \omega}{\sin(\omega+\gamma)} & 0 & 0 \\ \hline -1 & -1 & -2 \cos \gamma & -1 \\ 0 & 0 & -a & -a\frac{\sin \omega}{\sin(\omega+\gamma)} \end{array} \right) \quad (3.18)$$

where a is given by (2.10). The fact that the generators can never break links gives the Hamiltonian its lower block triangular structure. However it is important to note that the diagrammatic charge C does *not* commute with the 1BTL generators or with the Hamiltonian.

We can consider a ‘fake’ Hamiltonian \tilde{H}^{lp} which conserves diagrammatic charge (this Hamiltonian was used in [21]):

$$\tilde{H}^{lp} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -a\frac{\sin \omega}{\sin(\omega+\gamma)} & 0 & 0 \\ 0 & 0 & -2 \cos \gamma & -1 \\ 0 & 0 & -a & -a\frac{\sin \omega}{\sin(\omega+\gamma)} \end{pmatrix} \quad (3.19)$$

This of course has the same eigenvalues as H^{lp} but in general different eigenvectors.

The dimension of the eigenspaces of C (3.12) are the same as that of the S^z charge in the spin system if relate the eigenvalues m of S^z where $-\frac{L}{2} \leq m \leq \frac{L}{2}$ to the values of the diagrammatic charge in the following way:

- C even

$$m = \frac{L}{2} - C \quad (3.20)$$

- C odd

$$m = -\frac{L}{2} + C - 1 \quad (3.21)$$

In fact in Appendix E we prove using the Bethe ansatz that the eigenvalues of \tilde{H}^{lp} on the sector with S^z eigenvalue m coincide with the eigenvalues of H^d in the sector with the same value of m . This is illustrated in the final column of Tables 1 and 2 for the cases of $L = 2, 3$.

As we shall show in the next section the appearance of indecomposable structures related to ‘critical’ (non-semisimple) algebras can be understood in the case of H^{nd} . However in the link pattern representation the Jordan structures are unknown (see however Appendix B and Appendix D for the two site case). We have checked for different system sizes, by exact diagonalization, and found that for $\gamma = \omega = \frac{\pi}{3}$ there are no Jordan cell structures in H^{lp} . This implies that for this case H^{lp} can be brought by a similarity transformation to H^d . This special case is of importance since the Hamiltonian:

$$H^{stochastic} = a + L - 1 + H^{lp} \left(\gamma = \frac{\pi}{3}; \omega = \frac{\pi}{3} \right) \quad (3.22)$$

describes the time evolution of a fluctuating interface. The ground-state energy of $H^{stochastic}$ is zero for any number of sites and any value of $a \geq 0$ and has interesting combinatorial properties [32].

4 Properties of the centralizer of the 1BTL algebra in the ‘non-diagonal’ spin representation

4.1 Definition of the centralizer

We have described up to now three Hamiltonians corresponding to different representations of the 1BTL algebra (in the case of H^d this is more of a conjecture than a statement - see Appendix C). We have also repeatedly mentioned the possibility of the appearance of Jordan cells structures connected with the ‘critical’ algebras (2.5).

We have also to keep in mind the occurrence of degeneracies in the diagonal chain H^d [12–14].

In this section we are going to show that a centralizer of the 1BTL algebra in the representation in which the generators are given by (2.2) and (2.8) is going to bring a new insight in these problems. This centralizer was discovered by Doikou [29] but its properties and its relevance have not yet been studied, we are going to do it in this section.

For convenience, instead of the ferro-magnetic convention for the generators e_i used in (2.2), we will use the anti-ferromagnetic notation:

$$e_i = -\frac{1}{2} \left\{ \sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \cos \gamma \sigma_i^z \sigma_{i+1}^z - \cos \gamma + i \sin \gamma (\sigma_i^z - \sigma_{i+1}^z) \right\} \quad (4.1)$$

The expression of e_0 is also modified:

$$\begin{aligned} e_0 &= -\frac{1}{2} \frac{1}{\sin(\omega + \gamma)} (-i \cos \omega \sigma_1^z - \sigma_1^x - \sin \omega) \\ &= \frac{1}{2} \frac{1}{\sin(\omega + \gamma)} \begin{pmatrix} ie^{-i\omega} & 1 \\ 1 & -ie^{i\omega} \end{pmatrix} \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1} \end{aligned} \quad (4.2)$$

We have called this representation the ‘non-diagonal’ representation.

The quantum group $U_q(SU(2))$, is generated by $S^\pm, q^{\pm S^z}$ with the relations:

$$\begin{aligned} q^{S^z} S^\pm q^{-S^z} &= S^\pm \\ [S^+, S^-] &= \frac{q^{2S^z} - q^{-2S^z}}{q - q^{-1}} \end{aligned} \quad (4.3)$$

and co-products given by:

$$\begin{aligned} \Delta(S^\pm) &= q^{S^z} \otimes S^\pm + S^\pm \otimes q^{-S^z} \\ \Delta(q^{\pm S^z}) &= q^{\pm S^z} \otimes q^{\pm S^z} \end{aligned} \quad (4.4)$$

Using these we find that the action on the $SU(2)$ quantum spin chain is given by:

$$\begin{aligned} q^{\pm S^z} &= q^{\pm \frac{1}{2} \sigma^3} \otimes \cdots \otimes q^{\pm \frac{1}{2} \sigma^3} \\ S^\pm &= \sum_i q^{\frac{1}{2} \sigma^3} \otimes \cdots \otimes q^{\frac{1}{2} \sigma^3} \otimes \sigma_i^\pm \otimes q^{-\frac{1}{2} \sigma^3} \otimes \cdots \otimes q^{-\frac{1}{2} \sigma^3} \end{aligned} \quad (4.5)$$

As is well known [8], the generators of $U_q(SU(2))$, (with $q = e^{i\gamma}$) commute with the e_i ’s.

The generators S^z, S^+ and S^- do not commute however with e_0 . Doikou [29] has shown that in the ‘non-diagonal’ representation, the 1BTL algebra has a centralizer X :

$$[X, e_i] = 0 \quad [X, e_0] = 0 \quad (4.6)$$

which has the following expression:

$$X = \frac{1}{2 \sin(\gamma + \omega)} \left\{ e^{-\frac{1}{2}i\gamma} S^+ q^{-S^z} + e^{+\frac{1}{2}i\gamma} S^- q^{-S^z} - \frac{\cos \omega}{\sin \gamma} (q^{-2S^z} - 1) \right\} \quad (4.7)$$

where we have used the definition (4.3) and (4.4) of the quantum group generators. The normalization factor in (4.7) is, as it going to be seen, a very convenient one. One can see the expression of X as a ‘co-product’ form (very different from (4.4)):

$$\Delta(X) = 1 \otimes X + X \otimes q^{-2S^z} \quad (4.8)$$

It is clearly obvious that X commutes with the bulk Temperley-Lieb generators as it is constructed from $U_q(SU(2))$ quantum group generators. Using the co-product and the action on one site:

$$\begin{aligned} X_1 &= \frac{1}{2 \sin(\gamma + \omega)} \begin{pmatrix} \frac{ie^{-i\gamma/2} \cos \omega}{\cos \frac{\gamma}{2}} & 1 \\ 1 & \frac{-ie^{i\gamma/2} \cos \omega}{\cos \frac{\gamma}{2}} \end{pmatrix} \\ &= e_0 + \frac{\sin(\frac{1}{2}\gamma - \omega)}{2 \cos(\frac{1}{2}\gamma) \sin(\gamma + \omega)} \mathbf{1} \end{aligned} \quad (4.9)$$

one can easily see that it also commutes with the boundary generator e_0 .

We shall show in the next section that the centralizer X becomes indecomposable precisely in the ‘critical’ cases (2.5). This implies (see Appendix A) that degeneracies are induced in H^{nd} and implicitly in H^d and H^{lp} . The results of Appendix B for 2 sites and various numerical tests show that in the ‘critical’ cases H^{nd} always has the same indecomposable structure as X . In contrast the Hamiltonians H^{lp} and H^d , although having the same spectrum and degeneracies as H^{nd} , did not always have Jordan structure at the critical points. We found that the Hamiltonian H^{lp} had Jordan structure at some subset of the ‘critical’ points whereas H^d was always fully diagonalizable.

4.2 The spectrum and eigenfunctions of X : The Q-basis

We shall now calculate the eigenvalues and eigenvectors of the centralizer (4.7).

Firstly from diagonalizing the action of X on a one site system (4.9) we find its two eigenvalues can be written as:

$$\frac{\sin(-\frac{1}{2}\gamma) \sin(-\frac{1}{2}\gamma + \omega)}{\sin \gamma \sin(\gamma + \omega)}, \frac{\sin(\frac{1}{2}\gamma) \sin(\frac{1}{2}\gamma + \omega)}{\sin \gamma \sin(\gamma + \omega)} \quad (4.10)$$

We shall denote the operator X acting on an L site chain by $X^{(L)}$. Motivated by exact diagonalization at a low number of sites we found that the eigenvalues of $X^{(L)}$ took the following form:

$$\frac{\sin Q\gamma \sin(Q\gamma + \omega)}{\sin \gamma \sin(\gamma + \omega)} \quad \text{with degeneracy :} \quad \left(\frac{L}{2} - Q \right) \quad (4.11)$$

with $Q = -\frac{L}{2}, \dots, \frac{L}{2}$. The degeneracy given in (4.11) is for generic values of the parameters γ and ω . As we shall explain later this degeneracy becomes enhanced in the ‘critical’ cases (2.5). The one site results (4.10) are for $Q = -\frac{1}{2}, \frac{1}{2}$. The fact that the degeneracy is exactly that of an S^z operator suggests that we index the 2^L eigenvectors by the vector label $\mathbf{Q} = (Q_1; Q_2; \dots; Q_L)$ where $Q_i = \pm\frac{1}{2}$. The degeneracy is immediately explained if we take the eigenvalues to be:

$$\lambda^{(L)}(\mathbf{Q}) = \frac{\sin Q\gamma \sin(Q\gamma + \omega)}{\sin \gamma \sin(\gamma + \omega)} \quad \text{with : } Q = \sum_{i=1}^L Q_i \quad (4.12)$$

We shall now use the co-product of X (4.8) to prove this by induction and to find a general formula for the eigenvectors.

The eigenvectors $\mathbf{v}^{(L)}(\mathbf{Q}^{(L)})$ are vectors in the spin basis satisfying:

$$X^{(L)}\mathbf{v}^{(L)}(\mathbf{Q}) = \lambda^{(L)}(\mathbf{Q})\mathbf{v}^{(L)}(\mathbf{Q}) \quad (4.13)$$

We now form the most general eigenvectors of $X^{(L+1)}$. These are indexed by a $L+1$ dimensional vector $\mathbf{R} = (R_1; R_2; \dots; R_{L+1})$:

$$\mathbf{v}^{(L+1)}(\mathbf{R}) = \sum_{\mathbf{Q}} \left\{ a(\mathbf{R}, \mathbf{Q})\mathbf{v}^{(L)}(\mathbf{Q}) \otimes \uparrow + b(\mathbf{R}, \mathbf{Q})\mathbf{v}^{(L)}(\mathbf{Q}) \otimes \downarrow \right\} \quad (4.14)$$

Then using the co-product (4.8) we have:

$$\begin{aligned} X^{(L+1)}\mathbf{v}^{(L+1)}(\mathbf{R}) &= \sum_{\mathbf{Q}} \left\{ \mathbf{v}^{(L)}(\mathbf{Q}) \otimes \uparrow \left[a(\mathbf{R}, \mathbf{Q})A + b(\mathbf{R}, \mathbf{Q})B + q^{-1}a(\mathbf{R}, \mathbf{Q})\lambda^{(L)}(\mathbf{Q}) \right] \right. \\ &\quad \left. + \mathbf{v}^{(L)}(\mathbf{Q}) \otimes \downarrow \left[a(\mathbf{R}, \mathbf{Q})C + b(\mathbf{R}, \mathbf{Q})D + qb(\mathbf{R}, \mathbf{Q})\lambda^{(L)}(\mathbf{Q}) \right] \right\} \\ &= \lambda^{(L+1)}(\mathbf{R}) \sum_{\mathbf{Q}} \left\{ a(\mathbf{R}, \mathbf{Q})\mathbf{v}^{(L)}(\mathbf{Q}) \otimes \uparrow + b(\mathbf{R}, \mathbf{Q})\mathbf{v}^{(L)}(\mathbf{Q}) \otimes \downarrow \right\} \end{aligned} \quad (4.15)$$

where we abbreviate the action of X on a single site (4.9) by:

$$X \begin{pmatrix} \uparrow \\ \downarrow \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \uparrow \\ \downarrow \end{pmatrix} \quad (4.16)$$

Comparing terms we find:

$$\begin{aligned} \lambda^{(L+1)}(\mathbf{R})a(\mathbf{R}, \mathbf{Q}) &= a(\mathbf{R}, \mathbf{Q})A + b(\mathbf{R}, \mathbf{Q})B + q^{-1}a(\mathbf{R}, \mathbf{Q})\lambda^{(L)}(\mathbf{Q}) \\ \lambda^{(L+1)}(\mathbf{R})b(\mathbf{R}, \mathbf{Q}) &= a(\mathbf{R}, \mathbf{Q})C + b(\mathbf{R}, \mathbf{Q})D + qb(\mathbf{R}, \mathbf{Q})\lambda^{(L)}(\mathbf{Q}) \end{aligned} \quad (4.17)$$

Now we can eliminate $a(\mathbf{R}, \mathbf{Q}), b(\mathbf{R}, \mathbf{Q})$:

$$\left(\lambda^{(L+1)}(\mathbf{R}) - A - q^{-1}\lambda^{(L)}(\mathbf{Q}) \right) \left(\lambda^{(L+1)}(\mathbf{R}) - D - q\lambda^{(L)}(\mathbf{Q}) \right) = BC \quad (4.18)$$

Inserting the solution $\lambda^{(L)}(\mathbf{Q})$ valid for L sites (4.12) we find two solutions to (4.18) for $\lambda^{(L+1)}(\mathbf{R})$:

$$\lambda^{(L+1)}(\mathbf{R}) = \begin{cases} \frac{\sin((Q+\frac{1}{2})\gamma) \sin((Q+\frac{1}{2})\gamma+\omega)}{\sin \gamma \sin(\gamma+\omega)} \\ \frac{\sin((Q-\frac{1}{2})\gamma) \sin((Q-\frac{1}{2})\gamma+\omega)}{\sin \gamma \sin(\gamma+\omega)} \end{cases} \quad (4.19)$$

Therefore we see that:

$$\sum_{i=1}^{L+1} R_i = \sum_{i=1}^L Q_i \pm \frac{1}{2} \quad (4.20)$$

Now it is important to note that this constraint still allows considerable flexibility in the definition of the vector $\mathbf{R} = (R_1; \dots; R_{L+1})$. In particular it can be satisfied by choosing $\mathbf{R} = (\mathbf{Q}; R_{L+1})$ with the notation that the first L components of \mathbf{R} are given by \mathbf{Q} and the final one by $R_{L+1} = \pm \frac{1}{2}$. The motivation is that the similarity of \mathbf{Q} to \mathbf{S}^z might extend to allow the eigenvectors to be written as tensor products. At this point however this should be regarded as an ansatz whose ultimate justification will come when we are able to use it to write down a complete set of linearly independent eigenvectors. With this choice the sum (4.14) reduces to a single term:

$$\begin{aligned} \mathbf{v}^{(L+1)}(\mathbf{Q}; R_{L+1}) &= a(\mathbf{Q}; R_{L+1}) \mathbf{v}^{(L)}(\mathbf{Q}) \otimes \uparrow + b(\mathbf{Q}; R_{L+1}) \mathbf{v}^{(L)}(\mathbf{Q}) \otimes \downarrow \\ &= \mathbf{v}^{(L)}(\mathbf{Q}) \otimes [a(\mathbf{Q}; R_{L+1}) \uparrow + b(\mathbf{Q}; R_{L+1}) \downarrow] \end{aligned} \quad (4.21)$$

where we have, by an abuse of notation, written $a(\mathbf{Q}; R_{L+1})$ for the previous $a(\mathbf{R}, \mathbf{Q})$ with $\mathbf{R} = (\mathbf{Q}; R_{L+1})$ and similarly for $b(\mathbf{Q}; R_{L+1})$.

In other words the eigenstates can be written in a *tensor product* form. This is an important simplification and allows us to write them in a general form. To see this let us return to (4.17) and, knowing $\lambda^{(L)}(\mathbf{Q})$ and $\lambda^{(L+1)}(\mathbf{R})$ from (4.12) and (4.19), solve for $a(\mathbf{Q}; R_{L+1})$ and $b(\mathbf{Q}; R_{L+1})$. Firstly, up to normalization of the eigenfunction, only the ratio of these two functions is significant:

$$\frac{a(\mathbf{Q}; R_{L+1})}{b(\mathbf{Q}; R_{L+1})} = \frac{\lambda^{(L+1)}(\mathbf{Q}; R_{L+1}) - D - q\lambda^{(L)}(\mathbf{Q})}{C} \quad (4.22)$$

Inserting the formulae for C, D defined in (4.16) and the eigenvalues (4.12) one gets the following very simple expression:

$$\frac{a(\mathbf{Q}; R_{L+1})}{b(\mathbf{Q}; R_{L+1})} = \begin{cases} ie^{-2i\gamma Q - i\omega} & R_{L+1} = +\frac{1}{2} \\ ie^{2i\gamma Q + i\omega} & R_{L+1} = -\frac{1}{2} \end{cases} \quad (4.23)$$

These can be combined into a single expression:

$$\frac{a(\mathbf{Q}; R_{L+1})}{b(\mathbf{Q}; R_{L+1})} = ie^{-4i\gamma Q R_{L+1} - 2i\omega R_{L+1}} \quad (4.24)$$

As the R.H.S. is finite we can choose the normalization $b(\mathbf{Q}, R_{L+1}) = 1$. Then we obtain:

$$\mathbf{v}^{(L+1)}(\mathbf{Q}, R_{L+1}) = v^{(L)}(\mathbf{Q}) \otimes [ie^{-4i\gamma Q R_{L+1} - 2i\omega R_{L+1}} \uparrow + \downarrow] \quad (4.25)$$

where $Q = Q_1 + \dots + Q_L$. This equation defines a recurrence relation for the eigenvectors and can easily be solved.

The final result for the eigenvectors is:

$$\begin{aligned} \mathbf{v}^{(L)}(\mathbf{Q}) &= [ie^{-2i\omega Q_1} \uparrow + \downarrow] \otimes [ie^{-4i\gamma Q_1 Q_2 - 2i\omega Q_2} \uparrow + \downarrow] \\ &\otimes [ie^{-4i\gamma(Q_1+Q_2)Q_3 - 2i\omega Q_3} \uparrow + \downarrow] \otimes [ie^{-4i\gamma(Q_1+Q_2+Q_3)Q_4 - 2i\omega Q_4} \uparrow + \downarrow] \dots \\ &\otimes [ie^{-4i\gamma(Q_1+Q_2+\dots+Q_{L-1})Q_L - 2i\omega Q_L} \uparrow + \downarrow] \end{aligned} \quad (4.26)$$

where $\mathbf{Q} = (Q_1, Q_2, \dots, Q_L)$. We shall refer to this as the \mathbf{Q} -basis. The completeness of this basis will be discussed in the next subsection.

We have verified at one and two sites, by explicit diagonalization, that the eigenvectors are indeed given by the above form. Explicitly they are:

- One site

$$\begin{aligned} Q = +\frac{1}{2} &: ie^{-i\omega} \uparrow + \downarrow \\ Q = -\frac{1}{2} &: ie^{i\omega} \uparrow + \downarrow \end{aligned} \quad (4.27)$$

These states are linearly independent except when $w = 0$.

- Two sites

$$\begin{aligned} \mathbf{Q} = (+\frac{1}{2}; +\frac{1}{2}) &: -e^{-i\gamma - 2i\omega} \uparrow\uparrow + ie^{-i\omega} \uparrow\downarrow + ie^{-i\gamma - i\omega} \downarrow\uparrow + \downarrow\downarrow \\ \mathbf{Q} = (+\frac{1}{2}; -\frac{1}{2}) &: -e^{i\gamma} \uparrow\uparrow + ie^{-i\omega} \uparrow\downarrow + ie^{i\gamma + i\omega} \downarrow\uparrow + \downarrow\downarrow \\ \mathbf{Q} = (-\frac{1}{2}; +\frac{1}{2}) &: -e^{i\gamma} \uparrow\uparrow + ie^{i\omega} \uparrow\downarrow + ie^{i\gamma - i\omega} \downarrow\uparrow + \downarrow\downarrow \\ \mathbf{Q} = (-\frac{1}{2}; -\frac{1}{2}) &: -e^{-i\gamma + 2i\omega} \uparrow\uparrow + ie^{i\omega} \uparrow\downarrow + ie^{-i\gamma + i\omega} \downarrow\uparrow + \downarrow\downarrow \end{aligned} \quad (4.28)$$

These states are linearly independent except when $\omega = -\gamma, 0, \gamma$.

The spectrum of X (4.11) has additional degeneracies in the cases when $\lambda^{(L)}(Q) = \lambda^{(L)}(Q')$ has non-trivial solutions. This implies that we have solutions to one, or both, of the equations:

$$(Q + Q')\gamma + \omega = \pi\mathbf{Z} \quad (4.29)$$

$$(Q - Q')\gamma = \pi\mathbf{Z} \quad (4.30)$$

As we shall see in the next subsection the first equation (4.29) is related to the ‘critical’ points of the 1BTL algebra. The role of the second one (4.30) which is typical when

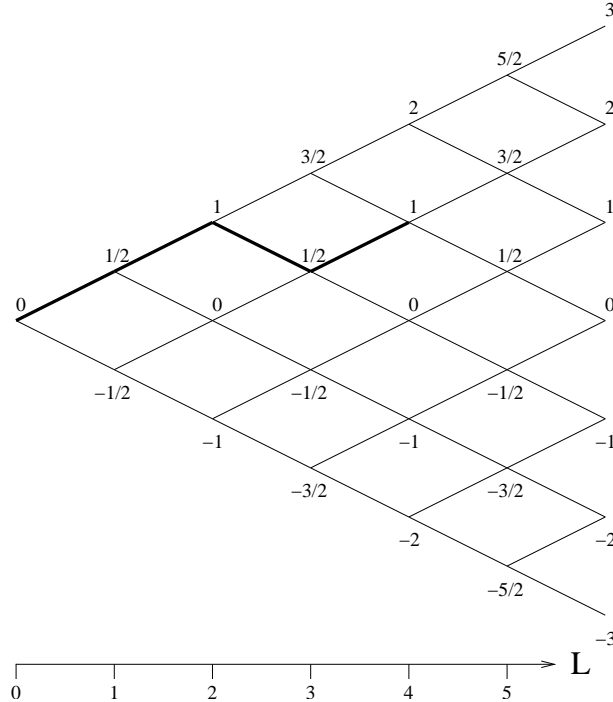


Figure 1: Full Bratelli Diagram. The system size, L , is given on the horizontal axis. The path corresponding to the eigenstate $\mathbf{v}^{(4)}(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$ is shown in bold.

a $U_q(SU(2))$ symmetry plays a role, is not obvious in the present context. Taking ω generic and $q = e^{i\gamma}$ a root of unity, one obtains degeneracies in the spectrum of X . However there are no Jordan cell structures in the expression of X . We would like to stress that the completeness of the \mathbf{Q} -basis (4.26) is not affected by (4.30). Simple examples suggest for q a root of unity there exist additional centralizers which do not have superfluous degeneracies. We hope to come back to this problem in another publication.

4.3 Truncated Bratelli diagrams and the critical points

The space of eigenvectors, with their values of \mathbf{Q} can be encoded in a Bratelli diagram (see Figure 1). From each different path on the diagram one reads off the values of Q_i and gets an eigenvector from (4.26). As there are two choices at each point ($Q_i = \pm \frac{1}{2}$) it is obvious that the space of solutions (4.26) has dimension 2^L . For a system of size L the degeneracy corresponding to a given value of Q is given by the binomial coefficient in (4.11). In terms of the Bratelli diagram it is the number of paths that start from the far left side at 0 and reach that point.

The set of eigenvectors for a system of size L may not always be linearly independent. The only problem in the construction of the basis is when the two possible expressions

in (4.23) coincide. In this case we have:

$$e^{-2i\gamma Q - i\omega} = e^{+2i\gamma Q + i\omega} \quad (4.31)$$

and therefore we have:

$$2\gamma Q + \omega = \pi \mathbf{Z} \quad (4.32)$$

This is the generalization of the results for 1 and 2 sites: (4.27) and (4.28). As Q takes integer and half-integer values (4.32) gives exactly the ‘critical’ cases of the one-boundary Temperley-Lieb algebra (2.5).

In these critical cases the basis breaks down. If from a particular state we can only produce one new eigenvector in (4.23) rather than two then it means that a complete basis of eigenvectors cannot be constructed - in other words the centralizer is becoming of Jordan form. In this case one can form a reduced space in which we simply remove all such states.

Let us illustrate with the example $\gamma = \frac{\pi}{5}, \omega = 2\pi/5$. The action of the centralizer on the states V_Q of different Q number is given by:

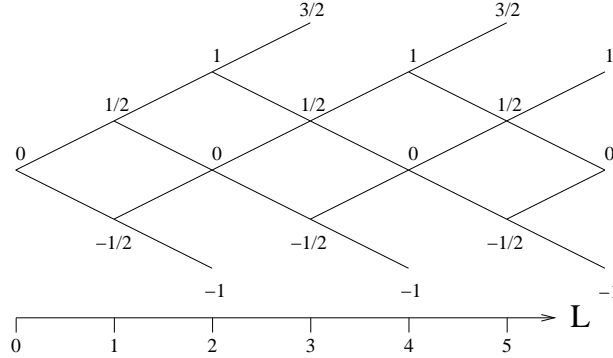
System size (L)	Value of Q								
	-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2
1	-	-	-	1	-	1	-	-	-
2	-	-	1	-	2	-	1	-	-
3	-	1*	-	3*	-	3	-	1	-
4	1*	-	4	-	6*	-	4**	-	1**
Eigenvalues of X	0	$\frac{-1}{\sqrt{5}}$	$\frac{1-\sqrt{5}}{2}$	$\frac{-1}{\sqrt{5}}$	0	$1 - \frac{1}{\sqrt{5}}$	1	$\frac{5+3\sqrt{5}}{10}$	1

In the cases in which the eigenvalues of the centralizer take different values in each Q sector there is a complete set of 2^L eigenvectors given by (4.26). However, as one can see from the above table, some of the different Q sectors give the same eigenvalues of the centralizer X and, by explicit diagonalization, one finds that different sectors begin to mix (indicated by * and **). At $L = 3$ we find that the 4 states from $Q = -\frac{3}{2}$ and $Q = -\frac{1}{2}$ combine into two eigenvectors and a single two dimensional Jordan cell. At $L = 4$ the $Q = -2$ and $Q = 0$ states form 5 eigenvectors and a single two dimensional Jordan cell; the $Q = 1$ and $Q = 2$ states form 3 eigenvectors and a single indecomposable representation.

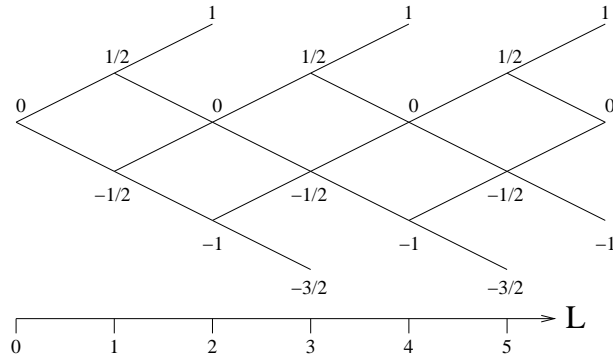
The fact that the centralizer cannot be completely diagonalized leads to degeneracy in the Hamiltonian H^{nd} (see Appendix A). However even for larger system sizes (we checked up to $L = 6$) we found that the indecomposable structures always broke into pairs and therefore the degeneracy in the Hamiltonian was never more than doublets. Using the spectral equivalence this gives us an understanding of the appearance of singlets and doublets in the diagonal chain H^d . This is interesting because in the numerical work on the diagonal chain in the case $\omega = -\gamma$ [12–14] the minimal models

were obtained by disregarding the doublets. We now have an understanding of this numerical prescription - in the ‘non-diagonal’ representation it simply corresponds to discarding the indecomposable sector of the 1BTL!

From the condition (4.32) we see that the first occurrence of indecomposable structure is for $Q = -1, \frac{3}{2}$. This is seen in the above table as these states begin to mix. We can try to truncate the space of states by removing *all* the states created from $Q = -1$ and $\frac{3}{2}$. This leaves us with the diagram:



On this truncated space the centralizer is completely diagonalizable. These states form the irreducible representations of the 1BTL algebra in the XXZ representation. The degeneracy of a particular state can be read off as before by counting the number of paths leading to that point. In the case of $\gamma = \frac{\pi}{5}, \omega = \frac{3\pi}{5}$ the truncated Bratelli diagram is given by:



The truncated Bratelli diagrams presented here were derived from the properties of the centralizer X . For a different approach, see the papers of Martin et al. [22–25].

4.4 Action of 1BTL generators in Q -basis

For generic values of the parameters the basis of eigenvectors (4.26) can be constructed and used to diagonalize the Hamiltonian. The Q -basis is written in terms of states:

$$|Q_1; Q_2 \cdots; Q_L\rangle \quad (4.33)$$

The expressions for these in terms of the spin basis are given in (4.26). As the action for e_0 and e_i are known in the spin basis we can work out their action on the \mathbf{Q} -basis. For e_0 we have:

$$\begin{aligned} e_0 \left| \frac{1}{2}; Q_2 \cdots; Q_L \right\rangle &= \frac{\sin w}{\sin(\omega + \gamma)} \left| \frac{1}{2}; Q_2 \cdots; Q_L \right\rangle \\ e_0 \left| -\frac{1}{2}; Q_2 \cdots; Q_L \right\rangle &= 0 \end{aligned} \quad (4.34)$$

For e_i we have:

$$\begin{aligned} e_i \left| \cdots; Q_{i-1}; \frac{1}{2}; \frac{1}{2}; Q_{i+2}; \cdots \right\rangle &= 0 \\ e_i \left| \cdots; Q_{i-1}; \frac{1}{2}; -\frac{1}{2}; Q_{i+2}; \cdots \right\rangle &= \alpha \left| \cdots; Q_{i-1}; \frac{1}{2}; -\frac{1}{2}; Q_{i+2}; \cdots \right\rangle \\ &\quad - \alpha \left| \cdots; Q_{i-1}; -\frac{1}{2}; \frac{1}{2}; Q_{i+2}; \cdots \right\rangle \\ e_i \left| \cdots; Q_{i-1}; -\frac{1}{2}; \frac{1}{2}; Q_{i+2}; \cdots \right\rangle &= -\beta \left| \cdots; Q_{i-1}; \frac{1}{2}; -\frac{1}{2}; Q_{i+2}; \cdots \right\rangle \\ &\quad + \beta \left| \cdots; Q_{i-1}; -\frac{1}{2}; \frac{1}{2}; Q_{i+2}; \cdots \right\rangle \\ e_i \left| \cdots; Q_{i-1}; -\frac{1}{2}; -\frac{1}{2}; Q_{i+2}; \cdots \right\rangle &= 0 \end{aligned} \quad (4.35)$$

where:

$$\begin{aligned} \alpha &= \frac{\sin(2\gamma\tilde{Q} + \omega + \gamma)}{\sin(2\gamma\tilde{Q} + \omega)} \\ \beta &= \frac{\sin(2\gamma\tilde{Q} + \omega - \gamma)}{\sin(2\gamma\tilde{Q} + \omega)} \end{aligned} \quad (4.36)$$

and both α and β depend on the previous \mathbf{Q} spins through $\tilde{Q} = Q_1 + Q_2 + \cdots + Q_{i-1}$ and therefore the e_i 's act non-locally.

Within the \mathbf{Q} -basis one can see from (4.34) and (4.35) that both e_0 and the e_i 's act within sectors of a given value of Q . This was to be expected as the \mathbf{Q} -basis is a basis for the centralizer X .

In the \mathbf{Q} -basis the highest weight vector $\left| \frac{1}{2}; \frac{1}{2}; \cdots; \frac{1}{2} \right\rangle$ and lowest weight vector $\left| -\frac{1}{2}; -\frac{1}{2}; \cdots; -\frac{1}{2} \right\rangle$ are eigenstates of all the 1BTL generators:

$$e_0 \left| \frac{1}{2}; \frac{1}{2}; \cdots; \frac{1}{2} \right\rangle = \frac{\sin w}{\sin(\omega + \gamma)} \left| \frac{1}{2}; \frac{1}{2}; \cdots; \frac{1}{2} \right\rangle \quad (4.37)$$

$$e_i \left| \frac{1}{2}; \frac{1}{2}; \cdots; \frac{1}{2} \right\rangle = 0 \quad (4.38)$$

$$e_0 \left| -\frac{1}{2}; -\frac{1}{2}; \dots; -\frac{1}{2} \right\rangle = 0 \quad (4.39)$$

$$e_i \left| -\frac{1}{2}; -\frac{1}{2}; \dots; -\frac{1}{2} \right\rangle = 0 \quad (4.40)$$

This implies that H^{nd} has at least two eigenvalues that are independent of L . In some ‘critical’ cases in which we have Jordan cell structures the highest and/or lowest weight state might not belong to the truncated \mathbf{Q} -basis.

4.5 Two simple cases

In this subsection we shall show that for two of the exceptional cases, namely $\omega = \pm\gamma$, the action of the ‘master’ Hamiltonian (2.6) has linear (or no) dependence on the parameter a in the truncated sector. These phenomena were first observed in numerical results [12]. The case $\omega = -\gamma$ was discussed more recently using completely different methods in [33].

- $\omega = -\gamma$

For $L > 1$ the truncated sector is made purely from states:

$$\left| -\frac{1}{2}; Q_2; \dots; Q_L \right\rangle \quad (4.41)$$

using the action of e_0 (4.34) given in the previous section we find:

$$H \left| -\frac{1}{2}; Q_2; \dots; Q_L \right\rangle = - \sum_{i=1}^{L-1} e_i \left| -\frac{1}{2}; Q_2; \dots; Q_L \right\rangle \quad (4.42)$$

and therefore the energy levels in the truncated sector do not depend on δ !

- $\omega = \gamma$

For $L > 1$ the truncated sector is made purely from states:

$$\left| \frac{1}{2}; Q_2; \dots; Q_L \right\rangle \quad (4.43)$$

using the action of e_0 given in the previous section (4.34) we find:

$$H \left| \frac{1}{2}; Q_2; \dots; Q_L \right\rangle = \left(-a \frac{\sin \gamma}{\sin(2\gamma)} - \sum_{i=1}^{L-1} e_i \right) \left| \frac{1}{2}; Q_2; \dots; Q_L \right\rangle \quad (4.44)$$

and therefore the energy levels in the truncated sector have only a linear dependence on a . In this sector the energy *differences* between the excited states and ground state are independent of δ .

These results are interesting because in [12] the spectrum of the Potts models with free boundary conditions was numerically found in the diagonal chain with $\omega = -\gamma$ by discarding the doublets and keeping the singlets. The energy of these singlet states did not depend on the value of δ .

5 Conclusions and open questions

The surprising result of this paper is that two XXZ Hamiltonians H^{nd} (2.9) and H^d (2.11), with the same bulk terms but very different boundary conditions, and one Hamiltonian H^{lp} defined in the vector space of link patterns (see Section 3), all have the same spectrum. The proof is given in Appendix E using the Bethe ansatz. An interesting result of these calculations is that a so-called ‘on the wrong side of the equator’ problem shows up in H^d , contrary to the periodic case [31].

All three Hamiltonians are obtained considering a ‘master’ Hamiltonian (2.6) defined in terms of the generators of the one-boundary Temperley-Lieb (1BTL) algebra (see (2.1) and (2.4)). This algebra depends on two parameters γ and ω . The third parameter δ which appears in the coefficients of the boundary terms of H^{nd} and H^d does not appear in the definition of the algebra but is related to a coefficient of the boundary generator in the ‘master’ Hamiltonian.

The three Hamiltonians H^{nd} , H^d and H^{lp} correspond to different representations of the 1BTL algebra. The representations used for H^{nd} and H^{lp} are known ones [5, 25, 34]. The existence of a representation which should give H^d , is less obvious. In the example of two sites, this representation is given in Appendix B. It is shown that the two generators commute with the local charge S^z (2.12). We conjecture that such a representation exists for any number of generators. This conjecture is not so far-fetched since, as is going to be shown in [30], a representation of the 1BTL algebra in which all the generators commute with S^z exists. In this representation the bulk generators have the standard form (2.2) but the boundary generator is non-local. What remains to be shown is that a similarity transformation brings us to the form (2.11) of H^d . So far we have only found this similarity transformation for small lattices.

The existence of ‘critical’ points, at which certain relations between the parameters γ and ω are satisfied (2.5) was pointed out by Martin et al. [22–25]. In these cases the 1BTL algebra possesses indecomposable representations. We have to stress that the indecomposable structures in the 1BTL algebra have nothing to do with those observed in the bulk Temperley-Lieb algebra.

The fact that the three Hamiltonians have the same spectra does not always imply that they are equivalent. This is due to the fact that they can possess different Jordan cell structures. In the simple example of two generators it is shown in Appendix B that there exist similarity transformations relating the three Hamiltonians at generic values of the parameters γ and ω . These transformations break down in cases in which one Hamiltonian has Jordan form but the other does not. These cases occur at (some subset of) the ‘critical’ points of the 1BTL. The question of whether a Hamiltonian in a given representation has Jordan cell form or is fully diagonalizable is linked to the question of the faithfulness of the representation (Appendix D).

In the case of H^{nd} , constructed using the ‘non-diagonal’ representation, a very good insight into the structure of indecomposable representations and degeneracies can be

obtained using the centralizer X (4.7) discovered by Doikou [29]. In this representation the centralizer commutes with the generators of the 1BTL. Although for generic values of γ and ω it is diagonalizable, in the ‘critical’ cases it possesses an indecomposable structure. We have computed the spectrum of X and found a basis in which X is diagonal (for generic values of γ and ω). This is the \mathbf{Q} -basis defined in section 4.2. In the \mathbf{Q} -basis, one can define an operator having the same eigenvalues (denoted by Q) and degeneracies as those seen in the spectrum of S^z . The generators of the 1BTL algebra conserve Q . If certain relations between ω and γ are satisfied (precisely the ‘critical’ cases found by Martin et al. [22–25]), X is not fully diagonalizable and the \mathbf{Q} -basis is not complete. It turns out the only Jordan cell structures that can appear are *two* dimensional. This observation is based on many numerical examples. We also show how to obtain a reduced vector space (without Jordan cells) using truncated Bratelli diagrams.

What did we learn from the existence of the centralizer of the ‘non-diagonal’ representation? Firstly in the generic cases H^{nd} can be block diagonalized using the eigenspaces of X . Secondly in the ‘critical’ cases the indecomposability of X implies (see Appendix A) that H^{nd} has degeneracies. One unexpected bonus is that, using the spectral equivalence, we get also get degeneracies in H^d and H^{lp} . In [12–14] it was observed, numerically, that for certain values of γ and ω , the spectrum of H^d is composed of doublets and singlets. This observation was used, with no deep reasons, to obtain the minimal models by disregarding the doublets. This ad hoc rule now gets an explanation. The singlets correspond to the ‘good’ reducible representations of the 1BTL algebra obtained using the truncated Bratelli diagrams described in Section 4.3 and the doublets correspond to the Jordan cells in H^{nd} . In section 4.5 we showed that for two of the ‘critical’ cases, namely $\omega = \pm\gamma$, the energy levels of the states in the truncated space had linear (or no) dependence on the boundary parameter a .

It was previously noticed in [7] that for a mysterious reason, in the continuum limit, the spectrum of H^d depends only on γ and ω and not on δ . The origin of the different role played by the three parameters is now clear. The continuum theory is dependent only on the two parameters γ and ω which belong to the 1BTL. The equality of the spectra implies, in particular, that in the finite-size scaling limit, the Coulomb gas description known for H^d [35] is valid also for the other two Hamiltonians.

If we limit our considerations to the XXZ chain with boundaries, we have seen that for two disconnected domains in the space of coefficients of the boundary terms (corresponding to H^{nd} and H^d) one can establish a spectral equivalence. For the decoupling point at $\Delta = 0$ using the results of [36] in the continuum limit one can show that the two domains exhaust the parameter space where equivalences to H^{nd} and H^d can be found.

The three representations of the 1BTL algebra considered here, namely those in H^{nd} , H^d , and H^{lp} , do not exhaust the possible 2^L dimensional representations. Another representation (depending on one free parameter) can be obtained using an ideal of the

two-boundary Temperley-Lieb algebra [21]. In this representation, similar to the link representation, one can define a diagrammatic charge and one can control the existence of Jordan cells through the free parameter. In [22–25] a faithful representation of 1BTL of dimension 2^L was given.

What is next? It is clear that one has to consider the ‘master’ Hamiltonian with two boundary generators (the two-boundary Temperley-Lieb algebra [21]). This will allow us to consider XXZ models with arbitrary boundary terms at the two ends of the chain and to define more integrable link pattern models. To find the Jordan cell structures in these cases will be a new exercise. In contrast to the 1BTL algebra, where the ‘critical’ values of the parameters are known, almost nothing is known about the 2BTL algebra.

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Appendix A Matrices with Jordan cells structure

In this paper matrices with Jordan cell structures appear in several different contexts. Here, using simple examples, we would like to mention two particular properties of such matrices.

Let X be a 2 by 2 matrix which is of the Jordan form:

$$X = \begin{pmatrix} x & 1 \\ 0 & x \end{pmatrix} \quad (\text{A.1})$$

It is trivial to check that if a matrix H commuting with X has the form:

$$H = \begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix} \quad (\text{A.2})$$

then this implies that the matrix H has a degenerate spectrum and can, but does not have to have, a Jordan form (one can take $\beta = 0$). This observation has an obvious application when X is a centralizer and H is the Hamiltonian. If X were not of Jordan form, then the condition that it commutes with H would not give rise to any degeneracies in the spectrum of the Hamiltonian. However if X is of the Jordan form

then the spectrum of H acquires a degeneracy even though there is only one operator which commutes with it.

We come now to the second observation that a Hamiltonian with Jordan structure leads to a different time evolution for certain observables. For example let us assume the Hamiltonian giving the Euclidean time evolution of a stochastic model [37] has the Jordan form given by the matrix H (A.2). For instance one has to compute the quantity

$$e^{-tH} |P\rangle = e^{-\alpha t} \begin{pmatrix} 1 & -\beta t \\ 0 & 1 \end{pmatrix} |P\rangle \quad (\text{A.3})$$

where $|P\rangle$ is the initial probability distribution. Notice that the time dependence is *not* purely exponential. This observation implies in particular that for systems described by Hamiltonians with Jordan cell structures the finite-size scaling limit behaviour of various quantities can be very different than the known ones.

Appendix B Similarity Transformations for $L = 2$ site Hamiltonians

The most general similarity transformation is:

$$UH^{nd} = H^d U \quad (\text{B.1})$$

which relates the two Hamiltonians H^{nd} (2.9) and H^d (2.11). For the two site case this is given by:

$$U = \begin{pmatrix} e^{i(\gamma+2\omega)}x_1 & ie^{i\omega}x_1 & -ie^{i(\gamma+\omega)}x_1 & x_1 \\ e^{-i\gamma}x_2 & A_{22} & A_{23} & x_2 \\ e^{-i\gamma}x_3 & A_{32} & A_{33} & x_3 \\ e^{i(\gamma-2\omega)}x_4 & ie^{-i\omega}x_4 & -ie^{i(\gamma-\omega)}x_4 & x_4 \end{pmatrix} \quad (\text{B.2})$$

where:

$$\begin{aligned} A_{22} &= -\{[\cos \delta \cot \gamma + \cos \omega(-i + \cot \gamma) + \sin \delta]x_2 + (\cos \delta + \cos \omega) \csc \gamma x_3\} \\ A_{23} &= -e^{-i\gamma} \csc \gamma [(\cos(\delta - \gamma) + e^{i\gamma} \cos \omega)x_2 + (\cos \delta + \cos \omega)x_3] \\ A_{32} &= -\{\csc \gamma(\cos \delta + \cos \omega)x_2 + [\cos \omega(-i + \cot \gamma) + \cos(\delta + \gamma) \csc \gamma]x_3\} \\ A_{33} &= -e^{-i\gamma} \{\csc \gamma(\cos \delta + \cos \omega)x_2 + [\cos \omega(i + \cot \gamma) + \cos(\delta + \gamma) \csc \gamma]x_3\} \end{aligned} \quad (\text{B.3})$$

and x_i , ($i = 1, \dots, 4$) are free parameters. From (B.2) we obtain:

$$\begin{aligned} \text{Det } U &= 4[\cos(2\gamma) - \cos(2\omega)] \csc \gamma \sin \omega x_1 x_4 \\ &\quad \left[-(\cos \delta + \cos \omega)x_2^2 + 2 \sin \delta \sin \gamma x_2 x_3 + (\cos \delta + \cos \omega)x_3^2 \right] \end{aligned} \quad (\text{B.4})$$

One can easily see that if

$$\omega = -\gamma, 0, \gamma \quad (\text{B.5})$$

there is no choice of the parameters x_i such that $\text{Det } U$ doesn't vanish. Therefore in these cases the similarity transformation does not exist. Notice that in (B.5) δ does not appear. This suggests that (B.5) is related to the 1BTL algebra and not specifically to the Hamiltonian. This is indeed the case, since these points are a special case of (2.5) which defines the 'critical' algebras.

The reason why the similarity transformation fails for the cases (B.5) is the appearance of Jordan cell structures. The eigenvalues of the two Hamiltonians are the same:

- $\omega = -\gamma$

$$\lambda_1 = -a \text{ (twice)}; \quad \lambda_2 = 0; \quad \lambda_3 = -2 \cos \gamma \quad (\text{B.6})$$

- $\omega = 0$

$$\lambda_1 = 0 \text{ (twice)}; \quad \lambda_2 = -\cos \gamma - \frac{1}{2}\sqrt{2 + 4a + 2 \cos(2\gamma)} \quad (\text{B.7})$$

$$\lambda_3 = -\cos \gamma + \frac{1}{2}\sqrt{2 + 4a + 2 \cos(2\gamma)} \quad (\text{B.8})$$

- $\omega = \gamma$

$$\lambda_1 = 0 \text{ (twice)}; \quad \lambda_2 = -\frac{a}{2 \cos \gamma}; \quad \lambda_3 = -2 \cos \gamma - \frac{a}{2 \cos \gamma} \quad (\text{B.9})$$

where a is defined in (2.10)

In all the exceptional cases (B.5) we find that H^{nd} has a Jordan cell structure:

$$H^{nd} \sim \begin{pmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_3 \end{pmatrix} \quad (\text{B.10})$$

while H^d is diagonalizable.

We now turn to the relation between H^{nd} and H^{lp} . We mention only the result. There exists a similarity transformation between the two Hamiltonians except for the special cases:

$$\omega = \pi/3, -\gamma. \quad (\text{B.11})$$

It turns out that for these cases H^{lp} is fully diagonalizable whereas H^{nd} is not. For the remaining cases in (B.5) both H^{lp} and H^{nd} are not fully diagonalizable and so can again be related by a similarity transformation.

Appendix C The ‘diagonal’ representation of the one-boundary Temperley-Lieb algebra

Using the representation given in equations (2.2) and (2.8) of the 1BTL algebra we have obtained H^{nd} given by (2.9). In Section 2, we mentioned the surprising observation that the spectrum of H^{nd} coincides with the spectrum of H^d given in (2.11). The latter Hamiltonian commutes with the local charge S^z (2.12). A natural question is: Does the 1BTL algebra have a representation in which each generator commutes with S^z ?

We consider here the two site problem and look for a 4×4 representation of the 1BTL algebra with the property:

$$[S^z, e_0] = 0 = [S^z, e_1] \quad (\text{C.1})$$

Such a representation exists:

$$e_0 = \begin{pmatrix} \frac{\sin \omega}{\sin(\omega+\gamma)} & 0 & 0 & 0 \\ 0 & \frac{\sin \omega}{\sin(\omega+\gamma)} & \cos\left(\frac{\omega+\delta}{2}\right) \sec\left(\frac{\omega-\delta}{2}\right) \frac{\sin \gamma}{\sin(\omega+\gamma)} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (\text{C.2})$$

$$e_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \eta & \eta\xi & 0 \\ 0 & 1 & \xi & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (\text{C.3})$$

where:

$$\begin{aligned} \eta &= \cos\left(\frac{\delta - 2\gamma - \omega}{2}\right) \sec\left(\frac{\delta - \omega}{2}\right) \\ \xi &= \cos\left(\frac{\delta + 2\gamma - \omega}{2}\right) \sec\left(\frac{\delta - \omega}{2}\right) \end{aligned} \quad (\text{C.4})$$

and δ is a free parameter. Notice that all parameters *including* δ appear in both e_0 and e_1 .

Taking a (which depends on δ) with the parameterization given in (2.10), we observe that the Master Hamiltonian (2.6):

$$H^M = -ae_0 - e_1 \quad (\text{C.5})$$

coincides with the XXZ Hamiltonian with diagonal boundary terms H^d (2.11) for the two site case.

Let us now perform a similarity transformation:

$$e_i \rightarrow e_i = U e_i U^{-1} \quad (i = 0, 1) \quad (\text{C.6})$$

with:

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \xi - e^{-i\gamma} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (\text{C.7})$$

then we obtain:

$$e_0 = \begin{pmatrix} \frac{\sin \omega}{\sin(\omega+\gamma)} & 0 & 0 & 0 \\ 0 & \frac{\sin \omega}{\sin(\omega+\gamma)} & 1 - \frac{e^{i\gamma} \sin \omega}{\sin(\omega+\gamma)} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (\text{C.8})$$

$$e_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & e^{i\gamma} & 1 & 0 \\ 0 & 1 & e^{-i\gamma} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (\text{C.9})$$

The similarity transformation (C.7) leaves S^z unchanged however it brings e_1 to the standard form (2.2) and removes the parameter δ from e_0 . The crucial difference between this ‘diagonal’ representation (C.8) and the ‘non-diagonal’ representation (2.8) of 1BTL is that e_0 acts now in the spin chain not only on site 1 but on the two sites 1 *and* 2. The representation of the 1BTL given in (C.8) generalizes [30] - the bulk generators e_i retain their standard form (2.2) and the generator e_0 now acts on *all* the L sites of the spin chain and commutes with S^z .

It is important to stress that the ‘non-diagonal’ and the ‘diagonal’ representations are equivalent except for the cases (2.5) when the 1BTL algebra is ‘critical’. In these ‘critical’ cases the ‘non-diagonal’ representation stays faithful whereas the ‘diagonal’ one is not (see Appendix D).

Appendix D Faithfulness of the representations

In this paper we have used several different representations of the 1BTL algebra. Although these are all of dimension 2^L we have seen that there can be a difference in the Jordan cell structures which appear. In this section we shall show that the question of possible Jordan cell structure in the Hamiltonians is intimately linked to their faithfulness. Here we shall only comment on the example of two sites.

At two sites using the 1BTL algebra we can construct only 6 words:

$$\mathbf{1}, e_0, e_1, e_1 e_0, e_0 e_1, e_0 e_1 e_0 \quad (\text{D.1})$$

If a representation is faithful for all values of the parameters γ and ω then there should *not* exist any linear relations between these words.

In the non-diagonal representation at all values of the parameters γ and ω we did not find any linear relations between the words and therefore we conclude that at $L = 2$ sites this representation is indeed faithful. It is possible that this is true in general but we have no proof.

In the diagonal and link pattern representations we again found that for generic values of the parameters γ and ω these were faithful. However for particular cases there were additional kernels in the representations. (For the case $\omega = -\gamma$ we rescale e_0 and have $e_0^2 = e_0$ and $e_1 e_0 e_1 = 0$.)

In the diagonal case the kernels are given by:

- $\omega = 0$

$$e_0 - e_0 e_1 e_0 = 0 \quad (\text{D.2})$$

- $\omega = -\gamma$

$$e_0 e_1 = 0 \quad (\text{D.3})$$

- $\omega = \gamma$

$$2 \cos \gamma e_1 e_0 - e_1 = 0 \quad (\text{D.4})$$

In the link-pattern case the kernels are given by:

- $\gamma = \frac{\pi}{3}, \omega = \frac{\pi}{3}$

$$e_1 e_0 - e_1 = 0 \quad (\text{D.5})$$

- $\omega = -\gamma$

$$e_0 e_1 = 0 \quad (\text{D.6})$$

These kernels are responsible for the disappearance of Jordan cell structure from the ‘master’ Hamiltonian (2.6). Let us illustrate this with an example $\omega = \gamma$. Using the 1BTL we find that the following is identically satisfied:

$$(H^M - \lambda_1)^2 (H^M - \lambda_2) (H^M - \lambda_3) = 0 \quad (\text{D.7})$$

where the λ_i were given in (B.9). This is equivalent to the statement that the ‘master’ Hamiltonian can be brought into the form:

$$H^M \sim \begin{pmatrix} \lambda_1 & c & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_3 \end{pmatrix} \quad (\text{D.8})$$

where c can take any value.

Now let us consider the case in which we can simplify (D.7) to:

$$(H^M - \lambda_1)(H^M - \lambda_2)(H^M - \lambda_3) = 0 \quad (\text{D.9})$$

This is equivalent to the statement that $c = 0$ in (D.8) i.e. the ‘master’ Hamiltonian (2.6) has no Jordan cell structure and can be completely diagonalized. Using just the one-boundary Temperley-Lieb algebra we find (D.9) is *not* an identity. Hence if the representation is faithful then the ‘master’ Hamiltonian cannot be fully diagonalized - as indeed observed for H^{nd} .

In the diagonal representation the L.H.S. of (D.9) is proportional to the kernel (D.4) and therefore does vanish. This implies that, in the diagonal representation, there is no Jordan cell structure in the ‘master’ Hamiltonian - as can be directly verified. All other differences of Jordan cell structure of Hamiltonians can be understood in a similar way.

Appendix E The spectra of H^{nd} , H^d and H^{lp} : The Bethe ansatz

In [21] the loop Hamiltonian H^{lp} was investigated and, as shown in Section 3, it has a block triangular form. In order to compute the spectrum it is sufficient to consider the block diagonal part which conserves the diagrammatic charge C . Any indecomposable structure that occurs between these block diagonal parts is simply ignored. One can write Bethe ansatz equations for this block diagonal part considering separately the case C even and C odd (see Tables 1 and 2 in Section 3). The reference states have $C = 0$ and $C = 1$ respectively. In [21] it was shown that the Bethe ansatz equations coincide with those obtained or conjectured in [17, 20] for the Hamiltonian H^{nd} . What remains to be shown is that the Hamiltonian H^d (2.11) has the same spectrum as H^{lp} .

We start (exactly as in [6]) with the Bethe reference state with all spins up and then add M down spins. We find the energy eigenvalues are given by:

$$E = - \sum_{i=1}^M (t + z_i + z_i^{-1}) \quad (\text{E.1})$$

where $t = 2 \cos \gamma$ and the z_i are solutions of:

$$z_i^{2L} = \frac{K(z_i)}{K(z_i^{-1})} \prod_{j=1}^M \frac{S(z_i^{-1}, z_j) S(z_j, z_i)}{S(z_j, z_i^{-1}) S(z_i, z_j)} \quad (\text{E.2})$$

with:

$$K(z) = t + z + z^{-1} - a(s + z) \quad S(z, w) = 1 + tw + zw \quad (\text{E.3})$$

and $s = \frac{\sin \omega}{\sin(\omega + \gamma)}$. If $M \leq \frac{L}{2}$ then this result exactly agrees with the eigenvalues of the loop Hamiltonian with C even. However one may also start in the spin chain from the reference state with all spins down and add M spin up waves. Then we find that in the sector with M spins up:

$$E = -as - \sum_{i=1}^M (t + z_i + z_i^{-1}) \quad (\text{E.4})$$

with the z_i given by:

$$z_i^{2L} = \frac{K(z_i)}{K(z_i^{-1})} \prod_{j=1}^M \frac{S(z_i^{-1}, z_j) S(z_j, z_i)}{S(z_j, z_i^{-1}) S(z_i, z_j)} \quad (\text{E.5})$$

where:

$$K(z) = t + z + z^{-1} + a(s + z(st - 1)) \quad S(z, w) = 1 + tw + zw \quad (\text{E.6})$$

This result now agrees for $M \leq L/2$ with the eigenvalues of the loop Hamiltonian with C odd. The first set of equations with $M > L/2$ does not give the same solutions as the second set with $M^* = L - M$. In fact, preliminary numerical studies indicate that the first set produces incorrect eigenvalues for such values of M . The complete spectrum is described by using both sets of equations and restricting $M \leq L/2$. We refer to [20] for similar numerical observations and [38] for related analytic results in the continuum limit.

The Bethe ansatz equations for the XXZ model with periodic boundary conditions has been investigated in detail in [31]. Again we can start from two different reference states: one with all spins up and the other with all spins down. Here to be ‘on the wrong side of the equator’, i.e. $M > \frac{L}{2}$, does not give rise to unresolvable problems and the Bethe ansatz solutions, including those ‘over the equator’, give the complete set of eigenstates.

The problem of being ‘on the wrong side of the equator’ can be seen most vividly in the case $st = 1$. Then one set of equations contains a manifest a dependence whereas the other does not. This case is interesting due to its relevance for stochastic processes at $t = 1, s = 1$ [32]. The complete set of wavefunctions is only obtained by combining those obtained from *both* Bethe reference states.

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